



GRETSA UNIVERSITY - THIKA

UNIVERSITY EXAMINATIONS JANUARY-APRIL 2025 SEMESTER

BACHELOR OF EDUCATION (ARTS)

COURSE CODE: BSCS 317

COURSE TITLE: LINEAR ANALYSIS

DATE: JANUARY 2025

TIME:

INSTRUCTIONS TO CANDIDATES

1. SECTION A IS **COMPULSORY**.
2. SECTION B: ANSWER ANY OTHER **TWO** QUESTIONS.
3. **DO NOT** WRITE ANYTHING ON THIS QUESTION PAPER AS IT WILL BE AN EXAM IRREGULARITY.
4. ALL ROUGH WORK SHOULD BE AT THE BACK OF YOUR ANSWER BOOKLET AND CROSSED OUT.

CAUTION: *All exam rooms are under CCTV surveillance during the examination period.*

SECTION A: COMPULSORY

Question One

- a) i) Give the meaning of a semi-norm. (2 marks)
- ii) Show that $p(x) = \sup_{n \geq 1} |x_n|$ is a semi norm on C , the space of all convergent sequences. (4 marks)
- b) Let C be the space of all convergent sequences. Define $\|x\| = \lim_{n \rightarrow \infty} |x_n|$. Show that $\|x\|$ is not a norm on C . (5 marks)
- c) Show that $T: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $T(x) = 2x - 1$ is not a linear operator. (4 marks)
- d) i) What do you understand by the term Metric. (4 marks)
- ii) In the set of $X = \mathbb{R}$ of all real numbers define on $\mathbb{R} \times \mathbb{R}$ by $d(x, y) = |x - y| \forall x, y \in \mathbb{R}$. Show that d is a metric on $\mathbb{R}$. (5 marks)
- e) If p_n is a sequence of paranorms in a linear space X , then $p(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x)}{1 + p_n(x)}$ is a paranorm on X . (6 marks)

SECTION B: ANSWER ANY TWO QUESTIONS

QUESTION TWO (20 MARKS)

- a) Let E be an inner product space, then for arbitrary vectors $x, y \in E$, $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$ with equality if and only if vectors x and y are linearly dependent. Prove. (6 marks)
- b) State without proof the Banach fixed point Theorem. (2 marks)
- c) State and prove the parallelogram law. (5 marks)

- d) i) Define a Hilbert space. (1 mark)
- ii) Show that the space L_p with $p \neq 2$ is not an inner product space and hence not a Hilbert. (6 marks)

QUESTION THREE (20 MARKS)

- a) Let T be a bounded linear operator. Show that the null space $N(T)$ or kernel of T is closed. (4 marks)
- b) Show that if $T: X \rightarrow X$ where X is a normed space is continuous at the origin, then T is uniformly continuous on X . (5 marks)
- c) i) Give the meaning of an inner product space (IPS). (4 marks)
- ii) Show that the linear space $\mathbb{R}^n$ (Euclidean space) with $\langle, \rangle$ defined for an arbitrary vectors $X = \{x_1, x_2, \dots, x_n\}$ and $y = \{y_1, y_2, \dots, y_n\}$ in $\mathbb{R}^n$ by $\langle, \rangle = \sum_{i=1}^n x_i y_i$ is an inner product space. (7 marks)

QUESTION FOUR (20 MARKS)

- a) Show that $f(x) = \int_a^b x(t) dt$ where $x(t) \in C[a, b]$ is a linear and bounded functional on $C[a, b]$. (7 marks)
- b) Let $T: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $T(x) = 7x$. Show that T is a linear operator. (3 marks)
- c) Define a complete metric space, hence show that the set $S = (0, 1]$ which is a subspace of $\mathbb{R}$ is not complete. (5 marks)

- d) Let $X=L_\infty$, the space of all bounded sequences. Define $p(x)=\sup_{n \geq 1} |x_n|$. Show that P is a paranorm. (5 marks)

QUESTION FIVE (20 MARKS)

- a) Let $T: D(T) \rightarrow Y$ be a linear operator where $D(T) \subset X$ and $R(T) \subset Y$. Prove that if T^{-1} exists, then it is a linear operator. (4 marks)
- b) Define a Banach space. (1 mark)
- c) Giving 2 examples give the meaning of a fixed point. (3 marks)
- d) i. Define a norm. (3 marks)
- ii. Show that $\|X\| = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}}$ is a norm ($1 \leq n \leq \infty$). (5 marks)
- e) A norm on the linear space X defines a metric on X by $d(x, y) = \|x - y\|$. Proof that d is induced by the norm. (4 marks)